INTRODUCING INVERSE FUNCTION TO HIGH SCHOOL STUDENTS: RELATING CONVENTION AND REASONING

Irma E. Stevens
University of Michigan
istevens@umich.edu

Inah Ko
University of Michigan
inahko@umich.edu

Teo Paoletti
University of Delaware
teop@udel.edu

Nicolas Boileau
University of Michigan
nboilea@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

Researchers have identified students' difficulties reasoning about inverse functions. Through our review of this literature, three meanings stand out: a formal, 'undoing', and quantitative meaning. Using these meanings as a guide, we analyzed student work collected from a lesson on the topic of inverse functions taught by an experienced high school mathematics teacher, using a novel task. In analyzing the data, we noticed tensions between students' understanding of the context and of inverse function as treated in curricula. In this paper, we illustrate these tensions and describe potential implications for students' productive construction of the meanings of inverse function.

Keywords: Cognition, Algebra, High School Education, Representation and Visualization

Previous literature has identified that students in high school and beyond struggle with constructing productive inverse function meanings. We identify three different ways researchers discuss students' meanings for inverse relations: formal, "undoing", and quantitative. In this study, we characterize student work from a contextualized, problem-based lesson (Herbst, 2003) that our research team co-designed with an experienced high school mathematics teacher to support students in developing productive inverse function meanings in relation to the meanings characterized in the literature. In particular, we designed the lesson to support students in conceiving of and representing a quantitative relationship. The teacher who taught the lesson stated as part of their goal that students would understand that a function and its inverse function represent the same relationship and that the rule used to determine the function could be "undone" to determine the rule for the inverse function. Addressing the question "How do students reason when introduced to inverse function?", we use examples of student work and dialogue during their discussion to characterize the extent to which students exhibited these meanings of inverse function. We also highlight how the teacher's attempt to meet institutional obligations (Chazan, Herbst, & Clark, 2016) by introducing switching techniques (described shortly) during the lesson likely prompted students to move away from their initial reasoning.

Prior Literature on Students' Meanings for Inverse Relationships and Framework

We synthesize three meanings for inverse functions that are emphasized in the research literature examining the learning and teaching of inverse function: a formal meaning, an 'undoing' meaning, and a quantitative meaning. We use these meanings to categorize both the students' work from the classroom and the teachers' discussion of inverse.

Formal Meaning

Many researchers characterizing students' and teachers' meanings of inverse functions have emphasized aspects of the formal definition of inverse function: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. This definition uses the notions of function composition (Brown & Reynolds, 2007; Even 1992; Vidakovic, 1996) and injectivity (Marmur & Zazkis, 2018; Wasserman, 2017). For example,

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Vidakovic (1996) provided a preliminary genetic decomposition of inverse function which closely resembled the formal definition. However, none of the students in her study developed inverse function meanings compatible with her genetic decomposition. Futhermore, none of the 26 pre-service teachers in Marmur and Zazkis' (2018) study noted the lack of injectivity of the function when asked to respond to a hypothetical student claiming the function $y = x^2 - 4x + 5$ had two inverse functions. The difficulties identified by Vidakovic (1996) and Marmur and Zazkis (2018) provide motivation for a continued need to explore ways to support students and teachers in developing meanings for inverse function.

Specifically, these aforementioned researchers and others (e.g., Paoletti et al., 2018) found that students' and teachers' meanings for inverse function are often constrained to engaging in specific actions in certain representations (e.g., switching-and-solving analytically, reflecting over a line graphically that may or may not result in equivalent inverse functions across these representations). For instance, Paoletti et al. (2018) noted a majority of the pre-service teachers in their study maintained disconnected meanings for inverse function that were constrained by such 'switching' techniques. Collectively, these disconnected meanings motivate a need to explore ways to support students and teachers in developing more coherent meanings for inverse function.

Undoing Meaning

Other researchers (Fowler, 2014; Martinez-Planell & Cruz Delgado, 2016; Oehrtman, Carlson, & Thompson, 2008; Teuscher, Palsky, & Palfreyman, 2018) have suggested having students develop meanings for an inverse function as "undoing" the original function process, often in lieu of focusing on formal mathematical properties of inverse function. Researchers who have adopted this stance have found that instruction emphasizing inverse functions as 'undoing' supports more students in addressing tasks relevant to decontextualized and contextualized inverse functions as compared to students who experienced instruction focused on formal definitions and switching techniques (e.g., analytically switching the x and y labels, reflecting over the line $y = x$). For example, across a sample of 3,858 college pre-calculus students, Teuscher, Palsky and Palfreyman (2018) reported that, in course sections with instruction emphasizing an undoing meaning, students accurately solved 48% of inverse function tasks compared to 32% of students whose instruction focused on switching techniques. We note that although emphasizing an 'undoing' meaning can be *more* productive when compared to emphasizing formal definitions or switching-techniques, over half of students in the former sections were still unsuccessful in addressing inverse function prompts.

Quantitative Meaning

Recently, Paoletti et al. (2018) and Paoletti (2020) have leveraged Thompson's (2011) theory of quantitative reasoning to characterize a quantitative meaning for inverse relations (and functions). A quantitative meaning for inverse relations entails a student understanding that a relation and its inverse relation represent an invariant relationship between quantities' values, regardless of how the relationship is represented. Thus, rather than foregrounding injectivity (cf. Marmur & Zazkis, 2018), a quantitative meaning entails the existence of an inverse relation regardless of whether the original or inverse represents a function. Students can determine if either relation is a function by examining if the univalence property (i.e., if for each value of one quantity there is exactly one value of the second quantity) holds for each relation.

Rather than focusing on a function and its inverse as processes that can be undone, a student with a quantitative meaning for inverse relations understands that a relation and its inverse are (or can be) represented by the same rule or graph (Paoletti, 2020). Paoletti (2020) provided an empirical example of one pre-service teacher reorganizing her unproductive inverse function meanings grounded in switching techniques, into a more productive, quantitative, meaning. By the end of the study the student, Arya, understood that a single graph or analytic rule represented a function and its inverse

function and that switching techniques were used to maintain conventions commonly used in school mathematics (e.g., the independent quantity is represented by variable x on the horizontal axis). She particularly noted how confusing switching techniques were in contextualized situations as it was necessary to switch the quantitative referents of the variables when engaging in switching techniques (i.e., if in $F(C) = (9/5)C + 32$, F represents the temperature in Fahrenheit and C the temperature in Celsius, then in $F^{-1}(C) = (5/9)(C - 32)$, F represents the temperature in Celsius and C the temperature in Fahrenheit). In this paper, we present indications of other students naturally maintaining the quantitative referents of variables.

Methods

Our team worked closely with an experienced high school mathematics teacher to design a contextual problem-based lesson with the goal of introducing students to the concept of inverse function. The first step in the lesson design was for the teacher to create a problem that would provoke a need for this new idea, but that students could make progress on by drawing on knowledge and skills that they had developed previously. Next, the teacher created a detailed lesson plan that included anticipations of student work and potential scaffolds and responses to them. The teacher then implemented the lesson in which students would work with their peers in small groups and then with whole-class discussions. The teacher ended the lesson with a statement of the newly introduced idea. The final version of the problem that the teacher designed is presented in Figure 1.

Several of us that teach at [name of your school] are on a slow-pitch recreation softball team together. Your City Parks and Rec charges a "sponsor fee" of $350 to enter the league. This pays for umpire fees, softballs, grounds people, etc. In addition, individual players each have to pay a player fee of $17. Thus, **the total amount of money we need to pay the office depends on how many people we have on our team**.

1) Make a table of *Total Fee* vs. *Number of Team Members* for at least 6 points. We need at least eight people to play.
2) Write, in words, the calculation procedure you kept doing to get the total amount of money given the number of players.
3) Is this situation linear? How do you know?
4) What is the y-intercept? What does it represent in this situation?
5) Write a rule for this situation.
6) Graph this function on a piece of graph paper.

When I worked for the recreation department in My Town, near the end of the season I needed to be able to see which teams in each division still had a chance to win the league. This way, I could order enough "Champions" t-shirts for the team with the most players who had a chance to win. What I had was the inventory list that had the receipts for the amount of money each team turned in, and from that, I had to figure out the number of players they had.

Assuming this same scenario for Your City Parks and Rec, the function for the league supervisor is backwards: for him or her, ***the number of players on the team depends on the total fee.***

7) Make a table for the league supervisor that computes the number of players for teams that have paid $571, $622, $639, $673 and $724.
8) Explain in words the calculation procedure you did to compute the number of players from the total fee amount.
9) Write a rule that computes the number of players as a function of the total team fee.
10) Make a second graph on your graph paper that shows the relationship from this perspective (with the total paid as the independent variable and the number of players the dependent).

11) Compare the function from the front with that of the recreation supervisor from this side. What do you notice about how the tables, graphs, and rules are different?

Figure 1: The Softball Fees Problem

Relative to the meanings for inverse function, although the context included an injective relationship between quantities and could have led to a discussion involving composition, this introductory lesson to inverse function did not explicitly address composition. Rather, the lesson revolved around ideas of inverse function more closely related to the undoing and quantitative meanings of inverse function. In the discussion of the lesson, the teacher explicitly referred to the inverse equation as representing an "undoing" of the original function process. Moreover, in his lesson plan, the teacher described, as a mathematical goal, that "we can focus at this initial stage on simply what the relationship looks like if you want to change your perspective and have students recognize that the function and its inverse are related, but not the same" (which is consistent with an undoing meaning) and goes on to say "that the factual information [this many players equates to this much money] stays consistent regardless of what perspective you have" (which is consistent with a quantitative meaning) The teacher attended to constructing different representations for the function and its inverse and emphasized the importance of understanding that the two functions represent the same "factual information" for all representations. Thus, in this paper, we report on student work that stemmed from instruction that emphasized both an "undoing" notion of inverse function (particularly when representing the relationship between the quantities as rules) and maintaining the quantitative relationship that a function and its inverse represent.

The teacher worked at a large Midwestern public high school. He taught the lesson during three class periods to three different classes of students. The data collected from each implementation of the lesson included video recordings positioned strategically across the room to capture students' work in groups. Additionally, researchers took fieldnotes and created copies of written work from 60 students.

Our analysis focused primarily on the student work. Initially, we analyzed it using Balacheff and Gaudin's (2010) conception framework. Through this analysis, we noticed variation in the representations used—tables, rules, and graphs—by students as well as in how they operated on them in the process of finding the inverse function. We also noticed that there was an association between the representations/operations used and the students' control structure. Reflecting on these findings, literature emphasizing students' inconsistencies in representations of inverse relationships (Paoletti et al., 2018), and our own observations of students' attempts to update their work to fit the conventional ways of representing inverse relationships in each of the representations (i.e., tables with the independent variable on the left, rules in which the independent variable is represented as *x*, and Cartesian graphs whose horizontal axis represents the independent variable *x*), we shifted our attention to those pieces of student work that may present this tension. We then open-coded (Strauss & Corbin, 1994) the student work based on individual student's ways of representing inverse functions (e.g., column location of values, naming of quantities in expressions, orientations of graphs). We also analyzed the classroom video associated with the implementations of the lesson to learn what ideas about inverse functions the teacher emphasized and to confirm the use of the aforementioned conventional notation with the students. From there, attending to the idea that conventional student work would not be present in work *prior* to the class discussion, we identified key pieces of student work in which the student seemed to change the way they represented the inverse function. As we explain in the next section, these pieces of work provided insights into students' initial reasoning about representing inverse functions and the alterations they made to fit with the ways in which the teacher was asking the students to represent inverse functions. We connected the chosen pieces of student work to ways in which they did so to the aforementioned

inverse function meanings. Collectively, this analysis allowed us to answer the question of how students reason when first introduced to inverse function and to identify some of the tensions that present in their different ways of representing their reasoning.

Results

We present samples of student work that provide evidence of reasoning emphasizing either an "undoing" or quantitative meaning for inverse function. Although each individual created tables, rules, and graphs, we only present specific responses relevant to specific representations of each student's work. Specifically, we describe, by type of representation, each piece of original student work and, if relevant, the updates that the students made to this work.

Reasoning with Tables

Addressing Question 7 (Figure 2), IS initially constructed a table for the supervisor that labeled the quantities "# of players" and "$" on the left and right, respectively. IS wrote the given fee values in the right-hand column, circling them. This table followed the same format as IS's first table from Question 1 (not pictured) in which IS had the number of players on the left side of the table and total fee amount on the right. To the right of that table in Figure 2 is an updated table in which the student constructed a table with switched columns (values and their associated quantitative referent). This example illustrates a student who considered their initial table as representing both a relationship and its inverse; using a single table to represent a function and its inverse aligns with the quantitative meaning for inverse function. Although students often chose different values or column labels in their tables, using the same table to address Question 1 and 7 was common. As a second example, MH preserved *x* and *y* labeling as well as the location of the quantities represented on the left and right in their table.

Figure 3 is another example of a student using the same table to address Questions 1 and 7. Moreover, KK's description of her original process for constructing her table across Questions 2 and 8 is consistent with an "undoing" meaning. Such activity may be indicative of the student understanding ways to connect her quantitative and undoing meanings for inverse function. She may understand that, while the function and its inverse represent the same relationship, in order to determine values of one quantity given a value of the second quantity, she must reverse the process by which she found values of the second quantity to determine the value of the first quantity. KK's table and descriptions also provide insight into the significant components of the table they were considering when updating their crossed-out table. Specifically, they drew several double-sided arrows on their (initial) crossed out table and, beside their redrawn table, they drew another blank table with the labels "ind" (independent) and "dep" (dependent). Comments from other students who drew new tables such as KK included a student writing "flipped around" above the new table and TK writing "should have put # of players right (y) on table." We conjecture such activity was spurned by the instructor who emphasized representing the independent variable on the left side of the table and the dependent variable on the right.

Lastly, consider the work in Figure 4 from DS. DS viewed a table as apt to represent both a relationship and its inverse. When asked to explain the calculation procedure for the supervisor, DS wrote, "You take the table from befor[e] and find the p[r]ice then writ[e] down what *x* is."

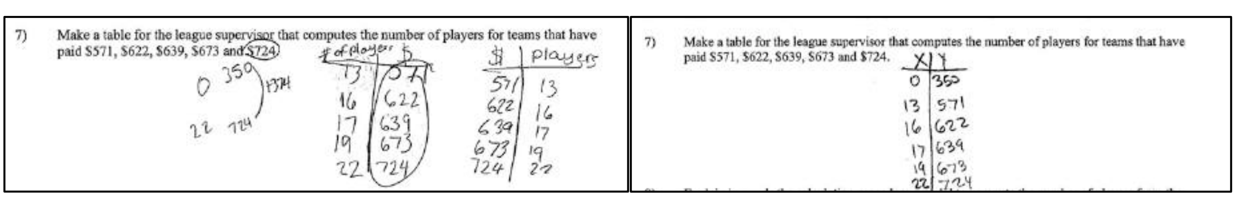

Figure 2: (left) IS's Two Tables for the League Supervisor and (right) MH's Table

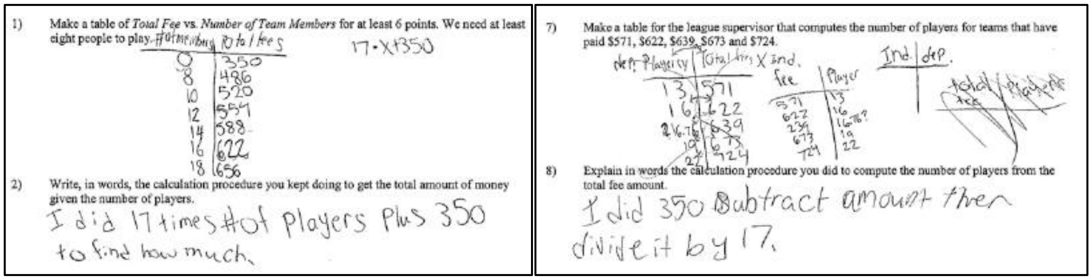

Figure 3: KK's Initial and Inverse Tables and "Undoing" Description

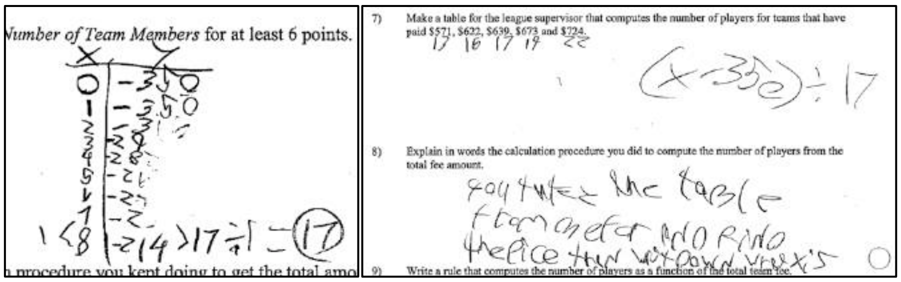

Figure 4: DS's Description of the Inverse Calculation Procedure

Reasoning with Rules

Figure 3 above provides an example of a student who provided a mathematical description of a process that undoes the original one. Perhaps due to current curricular treatments of inverse function, which emphasize the importance of representing the input quantity by the variable x on the horizontal axis, students engaging in writing a new rule to represent this undoing process may switch variables such that the independent quantity represents x and the dependent quantity represents y (or $f^{-1}(x)$). This point was raised by the teacher during the discussion. However, in students' initial work, there was variation among students' use of x, y, and their quantitative referents in their construction of a rule that computes the numbers of players as a function of the total team fee.

First—and indicative of maintaining a quantitative meaning for inverse functions—some students did not write a new rule for the inverse relationship and simply used their existing rule. For example, MH, who also did not construct a table with a different format for the inverse (Figure 2, right), used their rule "y = 17x+350" to substitute the given team fee values and solve for x (Figure 5). Thus, throughout their work on the task, x represents the "*number of team members*" and y represents the "*total fee*" consistently.

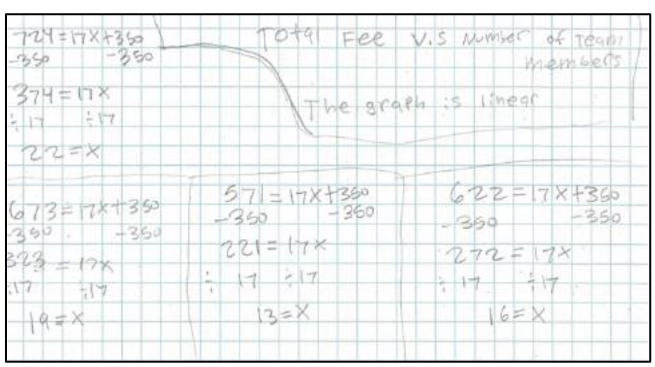

Figure 5: MH Substitution Strategy for Supervisor

Other students used their first rule and wrote a new rule in terms of the symbol representing the total team fee. For example, TK (Figure 6, left), who, like MH, did not construct a new table, had the rule "(y − 350) ÷ 17 = x". Here, y represented "Total fee" as labeled in the table. However, as seen in both MH's work and HV work, the x and y labels were not used consistently throughout individual students' work. For example, TK's initial rule, seen in Question 7 (i.e., (y – 350)/17 = x), maintains the quantitative referent of the variables in their table (i.e., "Total fee y"). However, when addressing Question 9, TK appears to have erased and then switched their original x and y labels in the rule used to create the table values. We conjecture this change may have been spurned by the classroom conversation based on TK's note that they "Should have put # of players right (y) on a table." Another student, HV, wrote the equation "(x − 350) ÷ 17 = y" but the labels on the table beside seem to indicate that x represented the number of players and y the total fee (Figure 6, right).

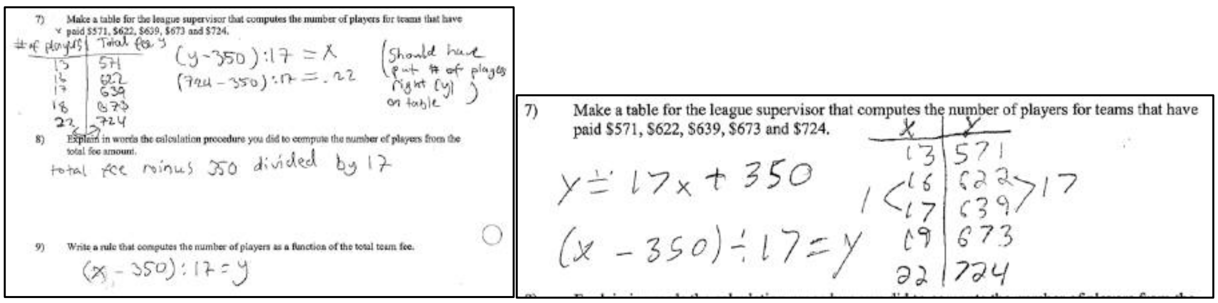

Figure 6: TK (left) and HV's (right) inconsistent use of *x* and *y*

Across the student work, we observed responses that were indicative of each of the meanings for inverse function described in the literature. Several students, like MH, exhibited a quantitative meaning for inverse function as they understood a single rule could be used to represent both a function and its inverse. Consistent with an undoing meaning for inverse function, other students, like TK, wrote a new rule that represented the opposite of the initial process that maintained the quantitative referents of the variables. Finally, several students created rules that inconsistently maintained the relationships between variables and quantitative referents, which may be indicative of their attempting to make sense of the classroom instruction that emphasized the importance of switching-and-solving.

Reasoning with Graphs

As a closing illustration, although most students constructed two perceptually different graphs with different axes labeled on the horizontal axis, nine students drew graphs (or at least labeled axes) to indicate that both requested graphs would have the same axes labels in the same locations. MX, for example, had "People" labeled on the horizontal axis for her first graph and seemed to intend for the number of people on the team to be represented on the horizontal axis for her graph for the supervisor, too (Figure 7). Like the students who only constructed one table (e.g., IS), these students seemed to indicate that a single graph orientation could represent both a relationship and its inverse.

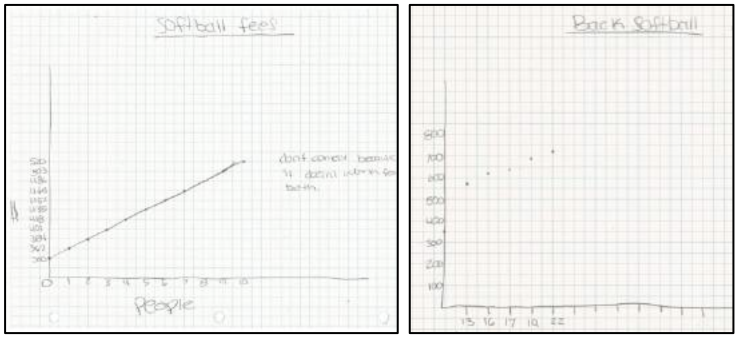

Figure 7: MX's Two Graphs with Same Axes Labels

Discussion

We use the aforementioned pieces of student work to highlight various ways in which students exhibited each of the three meanings for inverse relations present in the literature. The teacher designed the lesson purposefully to be contextualized and problem-based, and we argue reasoning with the context supported the students in understanding a relation and its inverse as representing the same quantitative relationships. The design of the task also supported an undoing meaning for inverse function. In particular, KK's work provides some evidence that these two meanings – quantitative and undoing meanings – can interplay with one another in possibly productive ways. We note that none of the student work contained the common struggles described in literature on students' inverse meanings (e.g., reflecting over a $y = x$ line on a graph, writing the multiplicative inverse of the function as the function's inverse, composing functions). We hypothesize this is because of the scaffolding of this introductory task and that the contextualization of the relationship they considered supported them in being able to reference the context to make sense of their results.

Despite the students providing several quantitative representations of their reasoning, the teacher likely felt an institutional obligation (Chazan, Herbst, & Clark, 2016) to carry out a classroom discussion in which students' reasoning, while perhaps quantitatively appropriate, needed to be amended to fit the conventional notations for inverse function. Although the teacher maintained consistent meanings for inverse function throughout this discussion, we note that students were faced with the tension of motivating changes in notation in their quantitatively appropriate work. We conjecture without having explicit conversations that allow students to reconcile the need for adjustments in their work (i.e. discussions regarding conventions), students may experience conflations (and perhaps a motivation to rely on memorizing techniques) when addressing inverse function tasks as is seen in the literature (e.g., Paoletti et al., 2018; Vidakovic, 1996). Future researchers may be interested in exploring how such conversations may be fruitful for teachers to have with students.

Acknowledgments

This work has been done with the support of grant 220020524 from the James S. McDonnell Foundation. All opinions are those of the authors and do not necessarily represent the views of the Foundation.

References

Balacheff, N. and N. Gaudin (2010). "Modeling students' conception: The case of function." *CBMS Issues in Mathematics Education*, 16: 183–211.

Brown, C. and B. Reynolds (2007). "Delineating Four Conceptions of Function: A Case of Composition and Inverse." *Conference Papers -- Psychology of Mathematics & Education of North America*: 1–193.

Chazan, D., Herbst, P., & Clark, L. (2016). Research on the teaching of mathematics: A call to theorize the role of society and schooling in mathematics instruction. *Handbook of research on teaching*. D. Gitomer and C. Bell. USA, American Educational Research Association: 1039-1097.
Even, R. (1992). The inverse function: Prospective teachers use of 'undoing'. *International Journal of Mathematics Education in Science and Technology*, *23*(4), 557–562.
Fowler, B. (2014). *An investigation of the teaching and learning of function inverse* (Master of Arts) Tempe, AZ: Arizona State University.
Herbst, P. (2003). Using Novel Tasks in Teaching Mathematics: Three Tensions Affecting the Work of the Teacher. *American Educational Research Journal, 40*(1), 197-238. Retrieved February 26, 2020, from www.jstor.org/stable/3699431
Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), *Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference* (Vol. 2, pp. 213–220). Mérida, México: UPN.
Marmur, O., & Zazkis, R. (2018) Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, *3*(99), 261–275.
Martinez-Planell, R., & Cruz Delgado, A. (2016). The unit circle approach to the construction of the sine and cosine functions and their inverses: An application of APOS theory. *The Journal of Mathematical Behavior*, *43*, 111–133.
Oehrtman, M., Carlson, M., & Thomson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding*. Making the connection: Research and teaching in undergraduate mathematics education*. M. P. Carlson and C. L. Rasmussen. Washington, D.C., Mathematical Association of America: 27–42.
Paoletti, T. (2020). Reasoning about relationships between quantities to reorganize inverse function meanings: The case of Arya. *The Journal of Mathematical Behavior*, *57*, 100741. Available at https://doi.org/10.1016/j.jmathb.2019.100741
Paoletti, T., Stevens, I. E., Hobson, N. L. F., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics*., 97(1), 93–109. Available at https://doi.org/10.1007/s10649-017-9787-y
Strauss, A., & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), Handbook of qualitative research (p. 273–285). Sage Publications, Inc.
Teuscher, D., Palsky, K., & Palfreyman, C. Y. (2018). Inverse functions: Why switch the variable? *Mathematics Teacher*, *111*(5), 374–381.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e.* S. Chamberlin, L. L. Hatfield and S. Belbase. Laramie, WY: University of Wyoming: 33-57.Vidakovic, D. (1996). "Learning the concept of inverse function." *Journal of Computers in Mathematics and Science Teaching*: 15(3): 295–318.
Vidakovic, D. (1996). "Learning the concept of inverse function." *Journal of Computers in Mathematics and Science Teaching* 15(3): 295–318.
Wasserman, N. (2017). Making sense of abstract algebra: Exploring secondary teachers' understanding of inverse function in relation to its group structure. *Mathematical Thinking and Learning*, *19*(3). 181–201.